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All quantum mixtures are what d’Espagnat has termed “improper.” His “proper” mixture cannot be created — if *welcher weg*, or distinguishing, information exists, an improper mixture results, while in the absence of such information, the resulting “mixture” is a pure state. D’Espagnat has claimed that an interpretation of the improper mixture in terms of subensembles leads to logical inconsistency; this claim is shown to be incorrect, as d’Espagnat’s argument fails to account for the indistinguishability of the pure-state subensembles.

## 1. Mixtures

The understanding of a statistical state as a mixture of subensembles of pure states is called an *ignorance interpretation* of the state. In his 1932 monograph, von Neumann made it clear that no such interpretation may be applied to the dispersion inherent in a pure quantum state (cf. Ref. 1, p. 302ff). However, in the same monograph, von Neumann used the ignorance interpretation in his introduction of mixtures exactly in the manner of classical probability. When introducing the statistical operator (p. 295), he says “if we do not even know what state is actually present — for example, when several states  $\phi_1, \phi_2, \dots$  with the respective probabilities  $w_1, w_2, \dots$  constitute the description,” then the statistical operator is  $\rho = \sum_s w_s |\phi_s\rangle\langle\phi_s|$ . Later, describing the measurement of an observable with eigenstates  $\{|p_j\rangle\}$  in a system in the unspecified state  $\rho_0$ , he says (p. 347) “the fraction  $\langle p_j | \rho_0 | p_j \rangle$  of the original ensemble ... is in the state  $|p_j\rangle$ ; ... [u]pon collecting these sub-ensembles, therefore, we obtain a mixture with the statistical operator”  $\rho$  (as above, with  $w_s = \langle p_s | \rho_0 | p_s \rangle$ ).

## 2. The propriety of mixtures

D’Espagnat<sup>(2-6)</sup> has introduced the category of “propriety” for mixtures:

The *proper* mixture is simply a mixture of subensembles of pure states  $|\phi_s\rangle$ , each with the weight  $w_s$ :  $\rho^S = \sum_s w_s |\phi_s\rangle\langle\phi_s|$  — exactly the mixture introduced by von Neumann.

The *improper* mixture refers to the trace-reduction of the statistical operator  $\rho^{S\oplus\mathcal{M}}$  of a composite system  $S \oplus \mathcal{M}$ :  $\rho^S \stackrel{\text{def}}{=} \text{Tr}_{\mathcal{M}}\{\rho^{S\oplus\mathcal{M}}\} = \sum_s w_s |\phi_s\rangle\langle\phi_s|$ . (This is pure only if  $\rho^{S\oplus\mathcal{M}} = \rho^S \otimes \rho^{\mathcal{M}}$  — i.e., only if  $S$  is uncorrelated with its exterior.)

D’Espagnat claims that an ignorance interpretation of the improper mixture is mathematically inconsistent, and concludes that the only candidate for state descriptor of a subsystem is inadequate. His proof consists in reconstructing the state of the composite system from the purported subensembles and obtaining a mixed state (a contradiction). This difficulty is illusory. In Sec. 4, I exhibit d’Espagnat’s argument and point out its error (which arises out of the neglect of indistinguishability). In Sec. 5, I show that no physical process can create a proper mixture.

## 3. Rules of distinguishability

The effect of *welcher weg* information was well-known and recognized from the very earliest days of quantum mechanics. Contrary to the early view that this effect is the result of irreducible disturbance due to observation, it has become clear that it is an intrinsic part of the formalism of quantum mechanics (cf., e.g., Refs. 7–9). The clearest statement (and perhaps most consistent use) of the principles regarding *welcher weg* distinguishability is that of Feynman.<sup>(10)</sup> Speaking *propagator* language, Feynman said (in paraphrase) “To find the probability of a process: when the alternative processes are indistinguishable, square the sum of their amplitudes; when distinguishable, sum the squares of their amplitudes.” Translating this into *state* language, we say “to construct the appropriate state representative of the system: when the alternative states are indistinguishable, add their vectors to get the state vector; when distinguishable, add their projectors to get the state operator.” (The vectors are weighted by the amplitudes of the processes leading to the alternative states, the projectors by the squares of these amplitudes).

## 4. D’Espagnat’s argument

D’Espagnat insists that the improper mixture, although represented by the same statistical operator as the proper mixture, does not represent a mixture of subensembles in pure states  $\{|\phi_s\rangle\}$ ; the ignorance interpretation may not be applied to it. The argument supporting this point is implicit in Feyerabend,<sup>(11)</sup> is explicit in d’Espagnat, and is frequently referred to in the literature of quantum interpretation. It is particularly clearly stated in Hughes (Ref. 12, p. 150; see also p. 283), whom I quote (taking some small liberties with the notation):

Consider a composite system in the *pure state*  $\rho^{S\oplus\mathcal{M}}$ , of which the component states are the mixed states  $\rho^S$  and  $\rho^{\mathcal{M}}$ . For the sake of argument, assume that  $\rho^S = a_1 |u_1\rangle\langle u_1| + a_2 |u_2\rangle\langle u_2|$ , while  $\rho^{\mathcal{M}} = b_1 |v_1\rangle\langle v_1| + b_2 |v_2\rangle\langle v_2|$ , with  $a_1 \neq a_2$  and  $b_1 \neq b_2$ , so there are no problems of degeneracy. Then, according to the ignorance interpretation of  $\rho^S$  and  $\rho^{\mathcal{M}}$ , system  $S$  is really in one of the pure states  $|u_1\rangle$  or  $|u_2\rangle$ , and system  $\mathcal{M}$  is really in one of the pure states  $|v_1\rangle$  or  $|v_2\rangle$ . ... But this would

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mean that the composite system is really in one of the four states  $|u_j v_k\rangle$ , with probabilities  $a_j b_k$  respectively — in other words, that the composite system is in a *mixed state*. Since this contradicts our original assumption, the ignorance interpretation simply will not do.

This argument is so clearly stated by Hughes that its error stands out: the claim that “the composite system is in a *mixed state*” is not supportable — nothing *external* to  $\mathcal{S} \oplus \mathcal{M}$  distinguishes those states  $|u_j v_k\rangle$  from one another. We must add the state vectors (not the projectors):  $|\Psi^{\mathcal{S} \oplus \mathcal{M}}\rangle = \sum_{jk} \psi_{jk} |u_j v_k\rangle$  — a pure state. The contradiction does not obtain; the argument fails to establish anything “improper” about these mixtures.

Hughes’s presentation is very close to that of Ref. 2 and of Ref. 3, p. 86. In Ref. 4, p. 61, d’Espagnat’s implicit assumption that any combining of pure-state subensembles yields a mixture is even clearer: he finishes with the explicit claim

... the fact that [the ensemble of the joint systems] should be describable as the union of all the [pure-state subensembles] implies that it should be describable also by a weighted sum of all the [pure-state projectors] ...

— exactly, and incorrectly, the rule given in the previous section for the combination of distinguishable alternatives.

## 5. The “proper” mixture cannot be created

How might we go about creating a mixture, in particular, a proper mixture? According to von Neumann’s description, echoed by d’Espagnat for the proper mixture, the preparation of the system  $\mathcal{S}$  varies randomly among the possible output states  $\{|\alpha_j\rangle\}$ . When  $\mathcal{S}$  is prepared in the state  $|\alpha_j\rangle$ , the state of its relevant environment<sup>1</sup>  $\mathcal{E}$  is  $|\eta_j\rangle$ , and the composite system  $\mathcal{S} \oplus \mathcal{E}$  is described by the state  $|\alpha_j \eta_j\rangle$ . The Indistinguishability Rule requires the state of  $\mathcal{S} \oplus \mathcal{E}$  to be the sum  $|\Psi^{\mathcal{S} \oplus \mathcal{E}}\rangle = \sum_s \gamma_s |\alpha_s \eta_s\rangle$ , hence pure.

If the  $\{|\eta_j\rangle\}$  are all collinear, the state of  $\mathcal{S}$  is  $|\Psi^{\mathcal{S}}\rangle = \sum_s \gamma_s |\alpha_s\rangle$  — pure, not a mixture! If the  $\{|\eta_j\rangle\}$  are not all collinear, then, utilizing the Schmidt decomposition, we have  $|\Psi^{\mathcal{S} \oplus \mathcal{E}}\rangle = \sum_s \psi_s |p_s a_s\rangle$ , with the  $\{|p_j\rangle\}$  and the  $\{|a_j\rangle\}$  orthonormal, and more than one non-vanishing coefficient<sup>2</sup>  $\psi_j$ ; then the state of  $\mathcal{S}$  is the *improper* mixture  $\rho^{\mathcal{S}} = \sum_s |\psi_s|^2 |p_s\rangle\langle p_s|$ . This analysis is exhaustive: it is not physically possible to create d’Espagnat’s proper mixture.

If  $\rho^{\mathcal{S}}$  is not pure, it may be expressed in infinitely many ways as a convex sum of linearly independent, but

<sup>1</sup> The “relevant environment” is the (smallest) system external to  $\mathcal{S}$  which is externally correlated with no system other than  $\mathcal{S}$ .

<sup>2</sup> It is always possible to destroy coherence (rather, its observability) by sloppy technique — the failure to observe fringes with an interferometer on a wobbly table is not interesting! All relevant experimental technique is assumed to have been applied to protect the observability of whatever coherence may be present — there are no coherence-destroying temporal random fluctuations in the coefficients. Cf. Ref. 9.

not necessarily orthogonal, projectors. According to the Ancilla Theorem<sup>(13)</sup> (see Appendix), all of — and only — these expressions are generated from  $|\Psi^{\mathcal{S} \oplus \mathcal{E}}\rangle$  by the unitary transformations on  $\mathcal{H}^{\mathcal{E}}$ : with the transformation  $|b_j\rangle = \sum_s \alpha_{js} |a_s\rangle$ , we have  $|\Psi^{\mathcal{S} \oplus \mathcal{E}}\rangle = \sum_s \phi_s |\phi_s b_s\rangle$  and  $\rho^{\mathcal{S}} = \sum_s |\phi_s|^2 |\phi_s\rangle\langle\phi_s|$ , where  $|\phi_j\rangle\langle\phi_j| = \sum_s \psi_s \alpha_{js}^* |p_s\rangle\langle p_s|$  and  $|\phi_j|^2 = \sum_s |\psi_j|^2 |\alpha_{js}|^2$ .

In summary, we have proven the physical existence of a system ancillary to any physical system in a mixed state:

**Theorem 1.** *If the state representative of a physical system  $\mathcal{S}$  is a mixture  $\rho^{\mathcal{S}} = \sum_s w_s |\phi_s\rangle\langle\phi_s|$ , then there must exist another physical system  $\mathcal{M}$  with an orthonormal basis  $\{ |b_s\rangle \in \mathcal{H}^{\mathcal{M}} \}$  such that  $|\Psi^{\mathcal{S} \oplus \mathcal{M}}\rangle = \sum_s \phi_s |\phi_s b_s\rangle$ , with  $|\phi_j|^2 = w_j$ .*

## 6. Discussion

The use of Hughes’s form of the argument is not a straw man against d’Espagnat, whose argument in Refs. 2–4 is essentially the same. In the more recent Refs. 5 and 6, d’Espagnat “simultaneously measures” the two systems to build up a mixture. But measurement is hardly an innocent activity in quantum mechanics: it creates the very distinguishability that does, in fact, produce the mixed state of the conclusion; in doing so, modifies the state of the joint system so it may no longer be assumed pure. Again, there is no contradiction.

For a recent discussion utilizing d’Espagnat’s distinction of propriety, consider Ref. 14, which gives as the “paradigmatic example” of a proper mixture the case of a preparation which depends on the result of a coin toss. But such a preparation does *not* lead to a proper mixture: The coin and the system are entangled; the system’s state is indeed a mixture, but improper — it is obtained by a partial trace over the coin’s space.

The error in d’Espagnat’s argument has “caused some confusion in the theory of measurement,” in Jammer’s phrase (Ref. 15, pp. 479-80, footnote), referring to the argument of Refs. 2 and 4) — but in the sense opposite to that meant by Jammer. The “confusion” referred to is the repeated rediscovery of what might be called the “Landau-collapse” analysis of quantum measurement (Ref. 16), in which tracing out the measuring apparatus leaves the system in the appropriate mixture, obviating von Neumann’s Process-1 collapse. Jammer implies that d’Espagnat’s argument has invalidated this understanding. But it is d’Espagnat’s argument which is incorrect; indeed, the subensembles described by the mixture are observable through their correlations with the ancillary system. The post-measurement  $\rho^{\mathcal{S}}$  mixture — exactly the result von Neumann requires (Ref. 1, p. 347) at the start of the argument which ends with the introduction of Process-1 (p. 351) — arises naturally, without use of Process-1.

### Related work of d’Espagnat

As the title of Ref. 6 would tell us, much of d’Espagnat’s interest surrounding this issue, and quantum mechanics in general, involves the issue of the determinacy of the val-

ues of variables: “realism.” Because I do not even mention this issue, much less discuss it at length, my criticism of his argument may seem rather “schematic”<sup>3</sup> compared with d’Espagnat’s discussions. But the application of the distinguishability heuristics to d’Espagnat’s ensemble argument requires no stance regarding the determinacy of variables.

Furthermore, as Examples 3 and 4 of Ref. 17 illustrate, indeterminate-valued variables occur naturally in irreducibly nondeterministic systems, systems which are perfectly ordinary, real, and entirely distinct from quantum mechanics. Value-indeterminacy is not a problem for the interpretation of quantum mechanics just so long as quantum mechanics is taken to be irreducibly nondeterministic. But in any case, none of the arguments of this paper depend on the issue of the determinism of quantum mechanics or the value-determinacy of its variables.

In a footnote on p. 1154 of Ref. 5, responding to another’s charge that the proper mixture does not exist, d’Espagnat claims that if the concept of a proper mixture is void, then the statement “immediately after an observable has been measured it *has* the observed value” is also a void statement, if the expression “it *has*” is to have its ordinary, commonsense meaning. I must disagree with d’Espagnat’s claim. Whatever one might mean by the “measurement” of an observable, it must minimally involve the correlation of the values of that observable with something outside the system. It follows directly, from basic rules of quantum mechanics, that, after this first stage of a measurement, the state of the system itself is an *improper* mixture. D’Espagnat’s proper mixture (whatever that might be) does not arise in measurement — the issue of its existence cannot therefore be the basis for any argument (physical or metaphysical) regarding measurement. This same error is seen in Section 3 of Ref. 18, where d’Espagnat considers a standard measurement process correlating the system with the pointer. He describes “the ensemble  $\mathcal{E}$  of all the pointers ... composed of  $\mu$  subensembles  $\mathcal{E}_1, \dots, \mathcal{E}_\alpha, \dots, \mathcal{E}_\mu$ , the components of each  $\mathcal{E}_\alpha$  being pointers really lying in one definite interval, the one labeled  $\alpha$ .” D’Espagnat then argues “since  $\mathcal{E}$  must be the addition of all the  $\mathcal{E}_\alpha$ , it is ... a ‘proper mixture’.” But each element in  $\mathcal{E}_\alpha$  is correlated with a corresponding element of the measured system, causing  $\mathcal{E}$  to be an *improper* mixture. (Curiously, earlier in that section d’Espagnat stated that  $\mathcal{E}$ ’s density matrix was obtained by partial tracing the system-plus-instrument density matrix — an improper mixture by, it would seem, his own standard.)

In the last paragraph of Section 3 of Ref. 18, d’Espagnat considers systems “that cannot be distinguished from one another by any measurement,” saying that, taking a realistic approach, “there is no reason why we should not consider ensembles of systems of the same type lying in different quantum states,” and claims that “there are cogent reasons to describe such ensembles by density matrices and call them mixtures.” But, as argued in Sec. 5 above, if these systems indeed cannot be distinguished, then an en-

semble of them will be described by a pure state operator — not a mixture, proper or improper.

## The “ignorance” interpretation of mixtures

That all mixtures are improper, thus correlated with ancillae, gives a clearer understanding of the temptation to their interpretation by ignorance. A mixed state of  $\mathcal{S}$ ,  $\rho = \sum_s w_s |\Phi_s\rangle\langle\Phi_s|$ , implies a correlation with an orthonormal basis (equivalently, with the values of some observable) of a system in the world exterior to  $\mathcal{S}$ ; the value of this ancillary observable identifies with certainty the state  $|\Phi_j\rangle$  of each occurrent  $\mathcal{S}$ , determined without an interaction with  $\mathcal{S}$ . The fact (as shown by the Ancilla Theorem) that *every* observable in the ancillary system has such a correlation with a distinct expansion of  $\rho$  as a convex sum of projectors does not introduce a problem with uniqueness: exactly *one* such ancillary observable may, in the occurrent actuality, be observed, thus exactly *one* such  $\mathcal{S}$ -mixture may be so identified.

## 7. Conclusion

D’Espagnat’s argument — concluding that it is inconsistent to consider as a mixture the state obtained by the trace-reduction of entangled systems — fails by failing to take into account the issue of indistinguishability. Further, fundamental rules of quantum mechanics prohibit the production of that which d’Espagnat calls a “proper” mixture. The category *proper mixture* is physically empty; states are either “improper” (correlated with ancillae) or pure (completely uncorrelated with the exterior).

The failure of d’Espagnat’s argument removes the only objection of substance (rather than of interpretive faith) to the Landau-collapse view of measurement decoherence.

## Appendix. The Ancilla Theorem

Here is a concise presentation of a theorem originally due to Gisin<sup>(19)</sup> and Hughston, Josza and Wootters,<sup>(20)</sup> with proof simplified by Mermin and others.<sup>(13)</sup> Mermin calls it “GHJW”; here, I refer to it by a less honorific but more descriptive name.

**Lemma.**  $|\Psi\rangle$  and  $|\Phi\rangle$  are vectors in  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . If  $\text{Tr}_2\{|\Psi\rangle\langle\Psi|\} = \text{Tr}_2\{|\Phi\rangle\langle\Phi|\}$ , then there exists a unitary transformation  $\mathbf{U}$  on  $\mathcal{H}_2$  such that  $|\Psi\rangle = (\mathbf{1} \otimes \mathbf{U})|\Phi\rangle$ .

*Proof.* Since it is positive and Hermitian,  $\text{Tr}_2\{|\Psi\rangle\langle\Psi|\} = \sum_j w_j |p_j\rangle\langle p_j|$ ,  $w_j > 0$  and  $\{|p_j\rangle \in \mathcal{H}_1\}$  orthonormal. Then, for any complete and orthonormal set  $\{|a_k\rangle \in \mathcal{H}_2\}$ ,  $|\Psi\rangle = \sum_{jk} \psi_{jk} |p_j a_k\rangle$ . Introducing the un-normalized kets  $|\beta_j\rangle \stackrel{\text{def}}{=} \sum_k \psi_{jk} |a_k\rangle$ , we have  $|\Psi\rangle = \sum_j |p_j \beta_j\rangle$ . Then  $\sum_j w_j |p_j\rangle\langle p_j| = \sum_{jj'} \langle\beta_j|\beta_{j'}\rangle |p_j\rangle\langle p_{j'}|$ ; thus  $\langle\beta_j|\beta_{j'}\rangle = w_j \delta_{jj'}$ , and we can write  $|\beta_j\rangle = \sqrt{w_j} |b_j\rangle$ , the  $\{|b_j\rangle\}$  orthonormal; hence  $|\Psi\rangle = \sum_j \sqrt{w_j} |p_j b_j\rangle$ . The same argument leads to  $|\Phi\rangle = \sum_j \sqrt{w_j} |p_j c_j\rangle$ . The sets  $\{|b_j\rangle\}$  and  $\{|c_j\rangle\}$  extended to orthonormal bases of  $\mathcal{H}_2$  are related by a unitary transformation  $|b_j\rangle = \mathbf{U}|c_j\rangle$  (explicitly,  $\mathbf{U} = \sum_s |b_s\rangle\langle c_s|$ ).  $\square$

<sup>3</sup> Professor d’Espagnat’s characterization (private communication).

### Theorem (Ancilla).

$|\Psi\rangle$  is a vector in  $\mathcal{H}_1 \otimes \mathcal{H}_2$ ;  $\rho_1 = \text{Tr}_2\{|\Psi\rangle\langle\Psi|\}$ . For any expression of  $\rho_1$  as a convex sum of distinct 1-projectors on  $\mathcal{H}_1$ ,  $\rho_1 = \sum_{j=1}^m f_j |\phi_j\rangle\langle\phi_j|$ ,  $m \leq \dim\mathcal{H}_2$ , there exists an orthonormal set  $\{|c_j\rangle\}$  in  $\mathcal{H}_2$  such that  $|\Psi\rangle = \sum_{j=1}^m \sqrt{f_j} |\phi_j\rangle \otimes |c_j\rangle$ .

*Proof.* Construct  $|\Psi'\rangle = \sum_{j=1}^m \sqrt{f_j} |\phi_j\rangle \otimes |d_j\rangle$ , with  $\{|d_j\rangle\}$  an arbitrary set of  $m$  orthonormal vectors in  $\mathcal{H}_2$ . By the Lemma, there exists a unitary transform  $\mathbf{U}$  on  $\mathcal{H}_2$  such that  $|\Psi\rangle = (\mathbf{1} \otimes \mathbf{U}) |\Psi'\rangle = \sum_{j=1}^m \sqrt{f_j} |\phi_j\rangle \otimes \mathbf{U} |d_j\rangle$ ;  $|c_j\rangle \stackrel{\text{def}}{=} \mathbf{U} |d_j\rangle$ .  $\square$

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